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LETTER TO THE EDITOR

New integrable differential-difference systems

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Abstract. We report a new, and relatively simple, procedure for finding new integrable differential-difference equations. The procedure starts from a rather general differential-difference equation taken in bilinear form and then searches for appropriate Bäcklund transformations for it. In this way three particular new integrable differential-difference systems are found and their corresponding Bäcklund transformations presented.

Hirota's bilinear method has been of fundamental importance to the theory of solitons since its inception [1–5]; and it has retained its significance throughout all of the subsequent developments [5, 6]. It is also the most direct and yet elementary approach for constructing exact multi-soliton solutions (cf [1–5, 7–9]). More recently the method has been systematically used in the search for new integrable equations in both $(1+1)$ and $(2+1)$ dimensions by finding equations with 3-soliton, 4-soliton, and even N-soliton solutions (cf [10–13]); as a result of such systematic tests some new examples of integrable equations have actually been found. However, by comparison with the continuous case, similar procedures for detecting integrable systems among the discrete lattices are scarcely developed. The problem for these discrete cases is that conditions for an N-soliton solution now become much more difficult to check. In the continuous case the N-soliton conditions are just polynomial identities. For example, for the Korteweg–de Vries (KdV)-type equation

$$F(\mathrm{D}_x, \mathrm{D}_t) f \cdot f = 0 \tag{1}$$

in which F is a polynomial function of D_x, D_t, and satisfies the conditions

$$F(\mathrm{D}_x, \mathrm{D}_t) = F(-\mathrm{D}_x, -\mathrm{D}_t) \qquad F(0,0) = 0$$

the N-soliton condition is [8]

$$\sum_{\sigma=\pm 1} F\left(\sum_{i=1}^{n} \sigma_i p_i, \sum_{i=1}^{n} \sigma_i \Omega_i\right) \prod_{i>j}^{n} F(\sigma_i p_i - \sigma_j p_j, \sigma_i \Omega_i - \sigma_j \Omega_j)\sigma_i \sigma_j = 0 \qquad n = 1, 2, \ldots, N$$

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in which $\sum_{\sigma=\pm1}$ means the summation over all possible combinations of $\sigma_1 = \pm1$, $\sigma_2 = \pm1, \ldots, \sigma_n = \pm1$; D_x, D_t are Hirota's bilinear operators [7, 8] defined by

$$\mathrm{D}_x^m \mathrm{D}_t^n a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')|_{x'=x,t'=t}.$$

In this letter, we consider the following bilinear differential-difference equation:

$$F(\mathrm{D}_x, \mathrm{D}_t, \sinh(\alpha_1\mathrm{D}_n), \ldots, \sinh(\alpha_l\mathrm{D}_n)) f(n;x,t)\cdot f(n;x,t) = 0. \tag{2}$$

Here, F is an even polynomial in D_x, D_t, $\sinh(\alpha_1\mathrm{D}_n), \ldots,$ and $\sinh(\alpha_l\mathrm{D}_n)$, and l is a given positive integer; the α_i, $i = 1, 2, \ldots, l$, are different constants, and $F(0, 0, \ldots, 0) = 0$;

$$\exp(\delta\mathrm{D}_n)a_n \cdot b_n \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta).$$

It is then straightforward to verify that equation (2) always has the 2-soliton solutions

$$f(n) = 1 + \exp(\eta_1) + \exp(\eta_2) + A_{12}\exp(\eta_1+\eta_2)$$

where

$$\eta_i = p_i n + q_i x + r_i t + \eta_i^0 \qquad F(q_i, r_i, \sinh(\alpha_1 p_i), \ldots, \sinh(\alpha_l p_i)) = 0 \qquad i = 1, 2$$

$$A_{12} = -\frac{F(q_1-q_2, r_1-r_2, \sinh(\alpha_1(p_1-p_2)), \ldots, \sinh(\alpha_l(p_1-p_2)))}{F(q_1+q_2, r_1+r_2, \sinh(\alpha_1(p_1+p_2)), \ldots, \sinh(\alpha_l(p_1+p_2)))}$$

where p_i, q_i r_i and η_i^0, for $i = 1, 2$, are constants.

However, when we want to go on to consider which specific forms of F admit 3-soliton solutions, the corresponding calculations become more involved. Accordingly, in the following, instead of searching for integrable systems by testing for N-soliton solutions, we propose a rather different scheme. Our procedure is to search for suitable F, G, A and B such that

$$\begin{aligned}
&[F(\mathrm{D}_x, \mathrm{D}_t, \sinh(\alpha_1\mathrm{D}_n), \ldots, \sinh(\alpha_l\mathrm{D}_n)) f(n)\cdot f(n)] \\
&\qquad \times[G(\mathrm{D}_x, \mathrm{D}_t, \sinh(\beta_1\mathrm{D}_n), \ldots, \sinh(\beta_{s_1}\mathrm{D}_n)) g(n)\cdot g(n)] \\
&\quad = [F(\mathrm{D}_x, \mathrm{D}_t, \sinh(\alpha_1\mathrm{D}_n), \ldots, \sinh(\alpha_l\mathrm{D}_n)) g(n)\cdot g(n)] \\
&\qquad \times[G(\mathrm{D}_x, \mathrm{D}_t, \sinh(\beta_1\mathrm{D}_n), \ldots, \sinh(\beta_{s_1}\mathrm{D}_n)) f(n)\cdot f(n)]
\end{aligned}$$

can be derived from

$$\begin{aligned}
&A(\mathrm{D}_x, \mathrm{D}_t, \exp(\gamma_1\mathrm{D}_n), \ldots, \exp(\gamma_{s_2}\mathrm{D}_n) f(n)\cdot g(n) = 0 \\
&B(\mathrm{D}_x, \mathrm{D}_t, \exp(\omega_1\mathrm{D}_n), \ldots, \exp(\omega_{s_3}\mathrm{D}_n) f(n)\cdot g(n) = 0
\end{aligned} \tag{3}$$

where s_i, $i = 1, 2, 3$, are given positive integers and β_i, $i = 1, \ldots, s_1$, γ_j, $j = 1, \ldots, s_2$, and ω_k, $k = 1, \ldots, s_3$, are constants. In this circumstance, (3) may be viewed as a Bäcklund transformation for (2) if $G(\mathrm{D}_x, \mathrm{D}_t, \sinh(\beta_1\mathrm{D}_n), \ldots, \sinh(\beta_{s_1}\mathrm{D}_n)) f(n)\cdot f(n) \neq 0$, a viewpoint which, by the introduction of F and G, seems somewhat more general than that of [7] for discovering Bäcklund transformations. Then, otherwise, *and in so far as Bäcklund transformations are characteristic of integrable systems*, equation (2) is an integrable system. Evidently it is relatively simple to find new integrable differential-difference systems in Hirota's bilinear form (2) by following this particular route.

In this letter we report *three* new integrable differential-difference equations found in this way.

Example 1. In this example we consider

$$[\mathrm{D}_t \sinh(\tfrac{1}{2}m\mathrm{D}_n) - 2\sinh(\tfrac{1}{2}k\mathrm{D}_n)\sinh(\tfrac{1}{2}(m-k)\mathrm{D}_n) + 2\alpha\sinh(\tfrac{1}{2}k\mathrm{D}_n)\sinh(\tfrac{1}{2}(m+k)\mathrm{D}_n)] \times f(n)\cdot f(n) = 0 \qquad (4)$$

where m and k are two integers and α is an arbitrary constant. In particular, when $\alpha = 0$, (4) becomes an extended Lotka–Volterra equation [14]. When $m = 1$, $k = 2$, (4) can be easily transformed into the generalized Lotka–Volterra equation found by Tsujimoto and Hirota [15]. We can show (4) is integrable in the sense of Bäcklund transformations. In fact we have the following.

Proposition 1. A Bäcklund transformation for (4) is

$$\exp(\tfrac{1}{2}(m-k)\mathrm{D}_n)f(n)\cdot g(n) = \left[\lambda\exp(\tfrac{1}{2}(m+k)\mathrm{D}_n) + \mu\exp(\tfrac{1}{2}(k-m)\mathrm{D}_n) + \frac{\alpha\mu}{\lambda}\exp(-\tfrac{1}{2}(m+k)\mathrm{D}_n)\right] \times f(n)\cdot g(n)$$

$$\left[\mathrm{D}_t - \lambda\exp(k\mathrm{D}_n) - \frac{\alpha}{\lambda}\exp(-k\mathrm{D}_n) - \gamma\right]f(n)\cdot g(n) = 0 \qquad (5)$$

where λ, μ and γ are arbitrary constants.

Remark 1. When $\alpha = 0$, the Bäcklund transformation (5) reduces to that in [16] for an extended Lotka–Volterra equation [14, 16].

Here, in the following, we derive some solutions of the bilinear equation (4) using the Bäcklund transformation (5). For simplicity, we set $\alpha = -1$. In this case, by applying the Bäcklund transformation (5) to the trivial solution $f(n) = 1$, we can easily obtain the following solutions:

(i) the 1-soliton solution

$$g(n) = 1 + \exp\{pn + 2\sinh(kp)t + \eta^0\}$$

where k, p and η^0 are constants, for the parameters $\lambda = \mathrm{e}^{kp}$, $\mu = -\lambda$, $\gamma = (1-\lambda^2)/\lambda$;

(ii) the rational solution

$$g(n) = n + 2kt$$

for the parameters $\lambda = 1$, $\mu = -1$, $\gamma = 0$.

Further by applying the Bäcklund transformation (5) to the 1-soliton solution $f(n) = 1 + \exp(\eta_1)$ we can deduce the following 2-soliton solution

$$g(n) = 1 + A_1\exp(\eta_1) + \exp(\eta_2) + A_2\exp(\eta_1+\eta_2)$$

where

$$\eta_i = p_i n + 2\sinh(kp_i)t + \eta_i^0$$

$$A_1 = \frac{\sinh(kp_1) + \sinh(kp_2) - \sinh[k(p_1+p_2)]}{\sinh(kp_1) - \sinh(kp_2) - \sinh[k(p_1-p_2)]} \qquad A_2 = -\frac{\sinh[\frac{1}{2}m(p_1-p_2)]}{\sinh[\frac{1}{2}m(p_1+p_2)]}$$

with p_i and η_i^0 constants, for the parameters $\lambda = \exp(kp_2)$, $\mu = -\lambda$ and $\gamma = (1-\lambda^2)/\lambda$.

Example 2. In this we consider an extended differential-difference KdV equation

$$[aD_z \sinh(\tfrac{1}{2}maD_n) + \sinh(\tfrac{1}{2}kaD_n)\sinh(\tfrac{1}{2}(m-k)aD_n)]f(n)\cdot f(n) = 0 \qquad (6)$$

$$[4a^3D_t \sinh(\tfrac{1}{2}maD_n) \pm aD_z \sinh((\tfrac{1}{2}m-k)aD_n) - 3\sinh(\tfrac{1}{2}kaD_n)\sinh(\tfrac{1}{2}(m-k)aD_n)] \times f(n)\cdot f(n) = 0 \qquad (7)$$

where z is an auxiliary variable and m, k are two integers. In particular, when $m = 1$, $k = -1$ and the minus sign is taken in (7), or $m = 1$, $k = 2$ and the plus sign is taken in (7), equations (6) and (7) become a differential-difference KdV equation proposed by Ohta and Hirota [17]. For this reason we call (6) and (7) an extended differential-difference KdV equation. Again we can show the system (6) and (7) is integrable. In fact we have the following.

Proposition 2. A Bäcklund transformation for equations (6) and (7) is

$$\exp(\tfrac{1}{2}(m-k)aD_n)f(n)\cdot g(n) = [\lambda\exp\{\tfrac{1}{2}(m+k)aD_n\} + \mu\exp\{\tfrac{1}{2}(k-m)aD_n\}] \times f(n)\cdot g(n) \qquad (8)$$

$$[2aD_z + \lambda\exp(kaD_n) + \gamma]f(n)\cdot g(n) = 0 \qquad (9)$$

$$[8a^3D_t + (-3\pm 1)\lambda\exp(kaD_n) \pm 2a\lambda D_z\exp(kaD_n) \pm \lambda\gamma\exp(kaD_n) + \omega] \times f(n)\cdot g(n) = 0 \qquad (10)$$

where λ, μ, γ and ω are arbitrary constants.

Remark 2. When $m = 1$, $k = 2$, the Bäcklund transformation equations (8)–(10) with the plus sign in (10) becomes one of the *two* Bäcklund transformations found in [18]. When $m = 1$, $k = -1$, the Bäcklund transformation equations (8)–(10) with the minus sign in (10) become another Bäcklund transformation for the differential-difference KdV equation. Note that as $a \to 0$ the differential-difference KdV equation becomes the KdV equation.

As an application of proposition 2, we derive 1-soliton and 2-soliton solutions of the extended differential-difference KdV equations (6) and (7). Here we only consider the case where the plus sign is taken in (7) and (10). The case where the minus sign is taken in (7) and (10) can be similarly considered. By applying the Bäcklund transformations (8)–(10) to the trivial solution $f(n) = 1$, we can easily obtain the following 1-soliton solution

$$g(n) = 1 + \exp(pn + qz + rt + \eta^0)$$

where

$$q = \frac{\lambda}{2a}(e^{-kap} - 1) \qquad r = \frac{\lambda}{4a^3}(1 - e^{-kap}) + \frac{\lambda^2}{8a^3}(1 - e^{-2kap})$$

with p and η^0 constants, for the parameters

$$\lambda = \frac{\sinh[\tfrac{1}{2}(m-k)ap]}{\exp(-\tfrac{1}{2}akp)\sinh(\tfrac{1}{2}map)} \qquad \mu = 1-\lambda \qquad \gamma = -\lambda \qquad \omega = 2\lambda + \lambda^2.$$

Furthermore, by applying the Bäcklund transformations (8)–(10) to the 1-soliton solution $f(n) = 1 + \exp(\eta_1)$, we can deduce the following 2-soliton solution

$$g(n) = 1 + \frac{\lambda_1 - \lambda_2\exp(kap_1)}{\lambda_1 - \lambda_2}\exp(\eta_1) + \frac{\lambda_1\exp(kap_2) - \lambda_2}{\lambda_1 - \lambda_2}\exp(\eta_2) + \frac{\lambda_1\exp(kap_2) - \lambda_2\exp(kap_1)}{\lambda_1 - \lambda_2}\exp(\eta_1 + \eta_2)$$

where

$$\eta_i = p_i n + q_i z + r_i t + \eta_i^0 \qquad \lambda_i = \frac{\sinh[\frac{1}{2}(m-k)ap_i]}{\exp(-\frac{1}{2}akp_i)\sinh(\frac{1}{2}map_i)}$$

$$q_i = \frac{\lambda_i}{2a}[\exp(-kap_i) - 1] \qquad r_i = \frac{\lambda_i}{4a^3}[1 - \exp(-kap_i)] + \frac{\lambda_i^2}{8a^3}[1 - \exp(-2kap_i)]$$

with p_i and η_i^0 constants, $i = 1, 2$, for the parameters

$$\lambda = \frac{\sinh[\frac{1}{2}(m-k)ap_2]}{\exp(-\frac{1}{2}akp_2)\sinh(\frac{1}{2}map_2)} \qquad \mu = 1 - \lambda \qquad \gamma = -\lambda \qquad \omega = 2\lambda + \lambda^2.$$

Example 3. In this example we consider a combined version of the two-dimensional Toda equation and a Lotka–Volterra-like equation:

$$[\mathrm{D}_x\mathrm{D}_t + A\mathrm{D}_t \sinh(\mathrm{D}_n) - 4\sinh^2(\tfrac{1}{2}\mathrm{D}_n)]f(n)\cdot f(n) = 0 \tag{11}$$

in which A is an arbitrary constant. In particular, when $A = 0$, (11) becomes the two-dimensional Toda equation [19]. If $f(n; x, t) \equiv f(n; t)$, then (11) becomes a special case of an extended Lotka–Volterra equation [14]. Concerning the more general equation (11) we have the following result.

Proposition 3. A Bäcklund transformation for (11) is

$$\mathrm{D}_x f(n)\cdot g(n) = \left[\lambda\exp(-\mathrm{D}_n) - \frac{A^2}{4\lambda}\exp(\mathrm{D}_n) + \mu\right]f(n)\cdot g(n) \tag{12}$$

$$[-4\lambda\mathrm{D}_t\exp(-\tfrac{1}{2}\mathrm{D}_n) + 2A\mathrm{D}_t\exp(\tfrac{1}{2}\mathrm{D}_n)]f(n)\cdot g(n)$$
$$= \left[\left(4 - \frac{A}{2\lambda}\gamma\right)\exp(\tfrac{1}{2}\mathrm{D}_n) + \gamma\exp(-\tfrac{1}{2}\mathrm{D}_n)\right]f(n)\cdot g(n) \tag{13}$$

where λ, μ and γ are arbitrary constants.

In the following, we derive 1-soliton and 2-soliton solutions of (11) using the Bäcklund transformation (12) and (13). For simplicity, we set $A = 2$. In this case, by applying the Bäcklund transformations (12) and (13) to the trivial solution $f(n) = 1$, we can easily obtain the following 1-soliton solution

$$g(n) = 1 + \exp(pn + qx + rt + \eta^0)$$

where

$$q = \lambda(1 - \mathrm{e}^p) + \frac{\mathrm{e}^{-p} - 1}{\lambda} \qquad r = \frac{\lambda}{1-\lambda}\frac{\mathrm{e}^p - 1}{\lambda\mathrm{e}^p - 1}$$

with p and η^0 constants, for the parameters $\mu = (1-\lambda^2)/\lambda$, $\gamma = 4\lambda/(1-\lambda)$. Furthermore, by applying the Bäcklund transformations (12) and (13) to the 1-soliton solution $f(n) = 1 + \exp(\eta_1)$, we can derive the following 2-soliton solution

$$g(n) = 1 + A_1\exp(\eta_1) + \exp(\eta_2) + A_2\exp(\eta_1 + \eta_2)$$

where

$$\eta_i = p_i n + q_i x + r_i t + \eta_i^0 \qquad q_i = \lambda_i[1 - \exp(p_i)] + \frac{\exp(-p_i) - 1}{\lambda_i}$$

$$r_i = \frac{\lambda_i}{1-\lambda_i}\frac{\exp(p_i) - 1}{\lambda_i\exp(p_i) - 1}$$

$$A_1 = \frac{(\lambda_1\lambda_2 - 1)[\lambda_1 - \lambda_2 \exp(p_i)]}{(\lambda_1 - \lambda_2)[\lambda_1\lambda_2 - \exp(p_i)]}$$

$$A_2 = \frac{\lambda_2(q_1 - q_2) + (1 - \lambda_2^2)\exp(-p_1 + p_2) + \lambda_2^2 - 1}{\lambda_2(q_1 + q_2) + (\lambda_2^2 - 1)\exp(p_1 + p_2) + 1 - \lambda_2^2}$$

with λ_i, p_i and η_i^0 constants, for the parameters $\lambda = \lambda_2$, $\mu = (1 - \lambda_2^2)/\lambda_2$ and $\gamma = 4\lambda_2/(1 - \lambda_2)$.

Propositions 1–3 can be proved by using bilinear operator identities. In the appendix, we give a proof in detail of proposition 2 as an illustrative example.

To summarize, in this letter a simple procedure for searching for integrable discrete systems in a bilinear form is described, and three new integrable differential-difference systems are reported. In each case the corresponding Bäcklund transformations are presented. Moreover, by using these Bäcklund transformations, we can derive further results obtaining exact solutions, such as soliton solutions and rational solutions. Thus, in particular, by starting from the Bäcklund transformation equations (8)–(10) as an example, we can derive the following result for equations (6) and (7).

Proposition 4. Let f_0 be a solution of the extended differential-difference KdV equations (6) and (7) and suppose that f_i, $i = 1, 2$, is a solution of equations (6) and (7), which is related by f_0 under the Bäcklund transformation equations (8)–(10) taken with parameters $(\lambda_i, \mu_i, \gamma_i, \omega_i)$, i.e. $f_0 \stackrel{(\lambda_i,\mu_i,\gamma_i,\omega_i)}{\longrightarrow} f_i$, $i = 1, 2$; $\lambda_1, \lambda_2, \mu_1, \mu_2 \neq 0$, and $f_j \neq 0$, $j = 0, 1, 2$. Then f_{12} defined by

$$\exp(\tfrac{1}{2}ka\mathrm{D}_n) f_0 \cdot f_{12} = c[\lambda_1 \exp(-\tfrac{1}{2}ka\mathrm{D}_n) - \lambda_2 \exp(\tfrac{1}{2}ka\mathrm{D}_n)] f_1 \cdot f_2 \qquad (14)$$

in which c is a non-zero constant, is a new solution of equations (6) and (7) which is related to f_1 and f_2 under the Bäcklund transformation equations (8)–(10) taken with parameters $(\lambda_2, \mu_2, \gamma_2, \omega_2)$, $(\lambda_1, \mu_1, \gamma_1, \omega_1)$, respectively.

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Appendix. Proof of proposition 2

First we list some bilinear operator identities:

$$\begin{aligned}
&[\mathrm{D}_y \sinh(\delta\mathrm{D}_n)a \cdot a][\exp(\delta\mathrm{D}_n)b \cdot b] - [\mathrm{D}_y \sinh(\delta\mathrm{D}_n)b \cdot b][\exp(\delta\mathrm{D}_n)a \cdot a] \\
&\qquad = 2\sinh(\delta\mathrm{D}_n)(\mathrm{D}_y a \cdot b) \cdot ab \qquad (\mathrm{A}.1)
\end{aligned}$$

$$\begin{aligned}
&[\mathrm{D}_y \sinh(\delta_1\mathrm{D}_n)a \cdot a][\exp(\delta_2\mathrm{D}_n)b \cdot b] - [\mathrm{D}_y \sinh(\delta_1\mathrm{D}_n)b \cdot b][\exp(\delta_2\mathrm{D}_n)a \cdot a] \\
&\qquad = \mathrm{D}_y \cosh(\tfrac{1}{2}(\delta_1 - \delta_2)\mathrm{D}_n)[\exp(\tfrac{1}{2}(\delta_1 + \delta_2)\mathrm{D}_n)a \cdot b] \cdot [\exp(-\tfrac{1}{2}(\delta_1 + \delta_2)\mathrm{D}_n)a \cdot b] \\
&\qquad\quad + \sinh(\tfrac{1}{2}(\delta_1 - \delta_2)\mathrm{D}_n)\{[\mathrm{D}_y \exp(\tfrac{1}{2}(\delta_1 + \delta_2)\mathrm{D}_n)a \cdot b] \cdot [\exp(-\tfrac{1}{2}(\delta_1 + \delta_2)\mathrm{D}_n)a \cdot b] \\
&\qquad\quad - [\exp(\tfrac{1}{2}(\delta_1 + \delta_2)\mathrm{D}_n)a \cdot b] \cdot [\mathrm{D}_y \exp(-\tfrac{1}{2}(\delta_1 + \delta_2)\mathrm{D}_n)a \cdot b]\} \qquad (\mathrm{A}.2)
\end{aligned}$$

$$\begin{aligned}
&[\sinh(\delta_1\mathrm{D}_n)\sinh(\delta_2\mathrm{D}_n)a \cdot a][\exp((\delta_1 + \delta_2)\mathrm{D}_n)b \cdot b] \\
&\qquad - [\sinh(\delta_1\mathrm{D}_n)\sinh(\delta_2\mathrm{D}_n)b \cdot b][\exp((\delta_1 + \delta_2)\mathrm{D}_n)a \cdot a] \\
&\qquad = \sinh(\delta_1\mathrm{D}_n)[\exp(\delta_2\mathrm{D}_n)a \cdot b] \cdot [\exp(-\delta_2\mathrm{D}_n)a \cdot b] \qquad (\mathrm{A}.3)
\end{aligned}$$

$$\sinh(\delta\mathrm{D}_n)a \cdot a = 0 \qquad (\mathrm{A}.4)$$

$$\begin{aligned} &\mathrm{D}_y \cosh(\delta_1 \mathrm{D}_n)[\exp((\delta_1+\delta_2)\mathrm{D}_n)a\cdot b]\cdot[\exp((\delta_1-\delta_2)\mathrm{D}_n)a\cdot b] \\ &\quad = \sinh(\delta_2\mathrm{D}_n)[\mathrm{D}_y \exp(2\delta_1\mathrm{D}_n)a\cdot b]\cdot ab \\ &\qquad + \sinh(\delta_2\mathrm{D}_n)(\mathrm{D}_y a\cdot b)\cdot[\exp(2\delta_1\mathrm{D}_n)a\cdot b]. \end{aligned} \tag{A.5}$$

We now turn to the proof of proposition 2. Let $f(n)$ be a solution of equations (6) and (7). If we can show that equations (8)–(10) guarantee that the following two relations hold,

$$P_1 \equiv [a\mathrm{D}_z \sinh(\tfrac{1}{2}ma\mathrm{D}_n) + \sinh(\tfrac{1}{2}ka\mathrm{D}_n)\sinh(\tfrac{1}{2}(m-k)a\mathrm{D}_n)]g(n)\cdot g(n) = 0$$

and

$$\begin{aligned} P_2 \equiv\ &[4a^3\mathrm{D}_t \sinh(\tfrac{1}{2}ma\mathrm{D}_n) \pm a\mathrm{D}_z \sinh((\tfrac{1}{2}m-k)a\mathrm{D}_n) \\ &-3\sinh(\tfrac{1}{2}ka\mathrm{D}_n)\sinh(\tfrac{1}{2}(m-k)a\mathrm{D}_n)]g(n)\cdot g(n) = 0 \end{aligned}$$

then equations (8)–(10) form a Bäcklund transformation.

In fact, in analogy with the proof already given in [16], we know that $P_1 = 0$ can be proved using equations (8) and (9). Thus it suffices to show that $P_2 = 0$. For this, by making use of (A.1)–(A.3), we have

$$\begin{aligned} -[\exp(\tfrac{1}{2}ma\mathrm{D}_n)f(n)\cdot f(n)]P_2 = &\ 8a^3\sinh(\tfrac{1}{2}ma\mathrm{D}_n)(\mathrm{D}_t f(n)\cdot g(n))\cdot f(n)g(n) \\ &-3\sinh(\tfrac{1}{2}ka\mathrm{D}_n)[\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\pm a\mathrm{D}_z\cosh(\tfrac{1}{2}ka\mathrm{D}_n)[\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\mp a\sinh(\tfrac{1}{2}ka\mathrm{D}_n)\{[\mathrm{D}_z\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &-[\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n)f(n)\cdot g(n)]\cdot[\mathrm{D}_z\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n)f(n)\cdot g(n)]\}. \end{aligned} \tag{A.6}$$

On the other hand, using the fact that $P_1 = 0$ and that $f(n)$ is a solution of (6), we know that

$$\begin{aligned} &[\exp((\tfrac{1}{2}m-k)a\mathrm{D}_n)g(n)\cdot g(n)] \\ &\qquad \times[a\mathrm{D}_z\sinh(\tfrac{1}{2}ma\mathrm{D}_n)+\sinh(\tfrac{1}{2}ka\mathrm{D}_n)\sinh(\tfrac{1}{2}(m-k)a\mathrm{D}_n)]f(n)\cdot f(n) \\ &\quad = [\exp((\tfrac{1}{2}m-k)a\mathrm{D}_n)f(n)\cdot f(n)] \\ &\qquad \times[a\mathrm{D}_z\sinh(\tfrac{1}{2}ma\mathrm{D}_n)+\sinh(\tfrac{1}{2}ka\mathrm{D}_n)\sinh(\tfrac{1}{2}(m-k)a\mathrm{D}_n)]g(n)\cdot g(n) \end{aligned}$$

and from this it follows, by using (A.2) and (A.3), that

$$\begin{aligned} &-a\sinh(\tfrac{1}{2}ka\mathrm{D}_n)\{[\mathrm{D}_z\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n)f(n)\cdot g(n)]\cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\qquad -[\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n)f(n)\cdot g(n)]\cdot[\mathrm{D}_z\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n)f(n)\cdot g(n)]\} \\ &\quad = a\mathrm{D}_z\cosh(\tfrac{1}{2}ka\mathrm{D}_n)[\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\qquad \cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\qquad +\sinh(\tfrac{1}{2}ka\mathrm{D}_n)[\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n)f(n)\cdot g(n)] \\ &\qquad \cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n)f(n)\cdot g(n)]. \end{aligned}$$

Thus by using equations (8)–(10), (A.4) and (A.5), we find that (A.6) becomes

$$
\begin{aligned}
-[\exp(\tfrac{1}{2}ma\mathrm{D}_n) f(n)\cdot f(n)]P_2
&= 8a^3 \sinh(\tfrac{1}{2}ma\mathrm{D}_n)(\mathrm{D}_t f(n)\cdot g(n))\cdot f(n)g(n)\\
&\quad +(-3\pm 1)\sinh(\tfrac{1}{2}ka\mathrm{D}_n)[\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n) f(n)\cdot g(n)]\\
&\quad \cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n) f(n)\cdot g(n)]\\
&\quad \pm 2a\mathrm{D}_z\cosh(\tfrac{1}{2}ka\mathrm{D}_n)[\exp(\tfrac{1}{2}(m-k)a\mathrm{D}_n) f(n)\cdot g(n)]\\
&\quad \cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n) f(n)\cdot g(n)]\\
&= 8a^3 \sinh(\tfrac{1}{2}ma\mathrm{D}_n)(\mathrm{D}_t f(n)\cdot g(n))\cdot f(n)g(n)\\
&\quad +(-3\pm 1)\lambda\sinh(\tfrac{1}{2}ka\mathrm{D}_n)[\exp(\tfrac{1}{2}(m+k)a\mathrm{D}_n) f(n)\cdot g(n)]\\
&\quad \cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n) f(n)\cdot g(n)]\\
&\quad \pm 2a\lambda\mathrm{D}_z\cosh(\tfrac{1}{2}ka\mathrm{D}_n)[\exp(\tfrac{1}{2}(m+k)a\mathrm{D}_n) f(n)\cdot g(n)]\\
&\quad \cdot[\exp(\tfrac{1}{2}(k-m)a\mathrm{D}_n) f(n)\cdot g(n)]\\
&= 8a^3 \sinh(\tfrac{1}{2}ma\mathrm{D}_n)(\mathrm{D}_t f(n)\cdot g(n))\cdot f(n)g(n)\\
&\quad +(-3\pm 1)\lambda\sinh(\tfrac{1}{2}ma\mathrm{D}_n)[\exp(ka\mathrm{D}_n) f(n)\cdot g(n)]\cdot f(n)g(n)\\
&\quad \pm 2a\lambda\sinh(\tfrac{1}{2}ma\mathrm{D}_n)[\mathrm{D}_z\exp(ka\mathrm{D}_n) f(n)\cdot g(n)]\cdot f(n)g(n)\\
&\quad \pm 2a\lambda\sinh(\tfrac{1}{2}ma\mathrm{D}_n)(\mathrm{D}_z f(n)\cdot g(n))\cdot[\exp(ka\mathrm{D}_n) f(n)\cdot g(n)]\\
&= 8a^3 \sinh(\tfrac{1}{2}ma\mathrm{D}_n)(\mathrm{D}_t f(n)\cdot g(n))\cdot f(n)g(n)\\
&\quad +(-3\pm 1)\lambda\sinh(\tfrac{1}{2}ma\mathrm{D}_n)[\exp(ka\mathrm{D}_n) f(n)\cdot g(n)]\cdot f(n)g(n)\\
&\quad \pm 2a\lambda\sinh(\tfrac{1}{2}ma\mathrm{D}_n)[\mathrm{D}_z\exp(ka\mathrm{D}_n) f(n)\cdot g(n)]\cdot f(n)g(n)\\
&\quad \pm\lambda\gamma\sinh(\tfrac{1}{2}ma\mathrm{D}_n)[\exp(ka\mathrm{D}_n) f(n)\cdot g(n)]\cdot f(n)g(n)\\
&= 0.
\end{aligned}
$$

In this way we have completed the proof of proposition 2.

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